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The Risk of War:

An Analysis Combining Real Options and Games

Laurent Gauthier*

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Abstract

Real options have been used to evaluate investment decisions with various structures, but have rarely been put into a game theoretical context. We examine the risk of war over a common resource, whose value is a geometric Brownian Motion, where one country may preemptively appropriate the resource, and the other wage war for it. We derive a closed form expression for the optimal mixed strategy that both countries should follow, and show that the resulting price level that triggers capturing the resource and subsequent war asymptotically follows a power law. The present value of the time until that trigger also decays slowly asymptotically. In consequence, in spite of both countries' propensity towards preemption, war may be indefinitely delayed. **Keywords:** Real options, resource wars, games, mixed strategies

1 Introduction

Trying to understand war and conflict from a rational perspective naturally raises all manners of paradoxes, one of the clearest being the famous adage “if you want peace, prepare for war”. The analysis of war in the mathematics of decision was initially tackled by Schelling ([1960] 1980) and Boulding (1962), with game theory and in a fairly static setup. As Maoz (1990) and Wagner (1983) showed, this approach leads to many paradoxes when one considers pure, or deterministic, strategies. Resource wars, in the whole gamut of human or even non-human conflict, have a specific trait in that one may be better able to frame the utility derived from waging war. Resources have value that may be readily identified, unlike other causes for

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war, such as honor, which are more difficult to assess Hirshleifer (1988). While economists have studied resource wars, such as Acemoglu et al. (2012), Caselli, Morelli, & Rohner (2015), Martin, Mayer, & Thoenig (2008) for example, these approaches do not seek to account for the particular dynamics of resource values.

In this paper, we rely on a real options framework in order to examine the risk of conflict around a resource accessible by two countries. This approach is comparable with investment decision in a competitive environment, to some extent. Real options are indeed useful in evaluating investment or disinvestment opportunities, but reflecting strategic behavior in a competitive environment can be complex, and potentially requires making specific assumptions about differences in firms or actors capabilities. For example, one may account for the differences between firms in a duopolistic situation through investment delays (Chesney & Gauthier 2006). In the case of the exploitation of a common resource, some models focus on realistically capturing the specifics of the competitive situation (D'Alpaos, Moretto, & Rosato 2023). Real options have been used to also model the decision to invest in war preparation (Medina & Nidier 2003). Nevertheless, in order to best account for the interactions of investment decisions in non-monopolistic cases, one needs to combine real options and games, an area in which there has been limited research (Grenadier 2000; Azevedo & Paxson 2014; Arasteh 2016). Further, these approaches mostly focus on pure strategies, especially for preemption.

In our approach, we will consider mixed (randomized) strategies, which can reach a broader optimum, while also offering an element of non-deterministic surprise, which is important in the context of war. This will require us to follow a probabilistic approach in studying the real options embedded in the model, because of the added complexity of optimal randomized strategies. This is distinct from usual approaches in real options where the optimal strategy is directly obtained at the optimum: we first express the present value of decision variables, and then optimize (Gauthier 2002a, 2002b). The second section presents the model and real option framework. The third section embeds the real options in a strategic framework and shows that the optimal distribution of the threshold at which one country will open the hostilities by seizing a commonly accessible resource asymptotically follows a power law, and that the time to conflict decays slowly.

2 A Real Option Model for Resource Appropriation

2.1 Modeling Assumptions

We consider two countries, indexed by $i \in \{1, 2\}$, and we will write $-i$ to designate “the other country” when we consider Country i . The market price of some resource is given by a stochastic process $(S_t)_{t \geq 0}$, a geometric Brownian Motion solution of: $dS_t = \mu dt + \sigma dB_t$. The stochastic process B is a standard Brownian Motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will assume that the drift is not too strong, that is $\mu < \frac{\sigma^2}{2}$. Each country needs to use this resource and acquires it on the market. The quantity of the resource in question, that must be consumed at each point in time, is a fixed amount ψ_i per unit of time. The total cost of using the resource at time t is therefore $\psi_i S_t$. Decisions are made using an actuarial logic, and both countries use the same discounting rate $\rho > \mu \vee 0$. Hence, for Country i , the expected cost of future resource procurement R^i at time 0 can be written:

$$R^i(S_0) = \mathbb{E}_{S_0} \left[- \int_0^\infty e^{-\rho s} \psi_i S_s ds \right] = -\psi_i \int_0^\infty e^{-\rho s} S_t \mathbb{E} \left[e^{(\mu - \frac{\sigma^2}{2})s + \sigma B_s} \right] ds = \frac{-\psi_i S_0}{\rho - \mu}.$$

Note that with our assumptions, $\rho - \mu$ is always positive. We will write $\alpha_i = \frac{\psi_i}{\rho - \mu}$, the present value of an infinite stream of resource cost growing at a rate of μ , so that $R^i(S_0) = -\alpha_i S_0$. R^i is a function of S_0 , and does not depend on time. For a cost C_i , let us assume Country i can appropriate territory where the resource can be found, in which case these resources do not need to be acquired in the market. Then, after this territory is occupied, the procurement cost becomes 0.

2.2 The Real Option Framework

If Country i is the only one who could appropriate the resource, then this decision corresponds to a standard real option. Since the cost only depends on the resource price S_t , the optimal decision threshold to appropriate the resource must be expressed as a hitting time of the form $T_h = \inf\{u \geq 0 : S_u = h\}$, with $h \geq S_0$. As a function of the threshold h , the total procurement cost, depending on the decision threshold h , can be written:

$$R_h^i(S_0) = \mathbb{E}_{S_0} \left[- \int_0^{T_h} e^{-\rho s} \psi_i S_s ds \right] - C_i \mathbb{E}_{S_0} [e^{-\rho T_h}]$$

In order to compute $\mathbb{E}_{S_0} \left[-\int_0^{T_h} e^{-\rho s} \psi_i S_s ds \right]$, we simply write it as:

$$\begin{aligned} \mathbb{E}_{S_0} \left[-\int_0^{T_h} e^{-\rho s} \psi_i S_s ds \right] &= \mathbb{E}_{S_0} \left[-\int_0^{\infty} e^{-\rho s} \psi_i S_s ds \right] - \mathbb{E}_{S_0} \left[-\int_{T_h}^{\infty} e^{-\rho s} \psi_i S_s ds \right] \\ &= \mathbb{E}_{S_0} \left[-\int_0^{\infty} e^{-\rho s} \psi_i S_s ds \right] - \mathbb{E}_{S_0} [e^{-\rho T_h}] \mathbb{E}_{S_{T_h}} \left[-\int_{T_h}^{\infty} e^{-\rho(s-T_h)} \psi_i S_s ds \right] \\ &= R^i(S_0) - \mathbb{E}_{S_0} [e^{-\rho T_h}] R^i(h). \end{aligned}$$

The last two equalities come from the fact that, due to the strong Markov property of the Brownian Motion applied at the stopping time T_h , the process starting from T_h has the same law as the process starting from h at any point in time. The Laplace transform of a geometric Brownian Motion hitting time is well known (Jeanblanc, Yor, & Chesney 2009, 152–153) and for $h \geq S_0$:

$$\mathbb{E}_{S_0} [e^{-\rho T_h}] = \left(\frac{S_0}{h} \right)^{\frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\frac{2\rho}{\sigma^2} + \left(\frac{1}{2} - \frac{\mu}{\sigma^2} \right)^2}}.$$

We will write $\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\frac{2\rho}{\sigma^2} + \left(\frac{1}{2} - \frac{\mu}{\sigma^2} \right)^2}$. Since $\mu < \frac{\sigma^2}{2}$ and $\rho > \mu$, we have $\beta > 1$ too. Hence, we obtain the value of the potential appropriation strategy $R_h^i(S_0)$:

$$\begin{aligned} R_h^i(S_0) &= R^i(S_0) - \mathbb{E}_{S_0} [e^{-\rho T_h}] R^i(h) - C_i \mathbb{E}_{S_0} [e^{-\rho T_h}] \\ &= -\alpha_i S_0 + \left(\frac{S_0}{h} \right)^\beta (\alpha_i h - C_i). \end{aligned}$$

This value can be maximized as a function of h to find the optimal strategy. Solving for $\frac{\partial R_h^i(S_0)}{\partial h} = 0$, we obtain: $h_i^* = C_i \frac{\beta}{\alpha_i(\beta-1)}$. The threshold expresses a ratio between the entry cost and the future cost of having to buy the resource. Finally, we can express the value of following the optimal strategy:

$$R_{h_i^*}^i(S_0) = \frac{C_i}{\beta-1} \left(S_0 \frac{\alpha_i(\beta-1)}{C_i\beta} \right)^\beta - \alpha_i S_0,$$

and we can see that $R_{h_i^*}^i(S_0) > R^i(S_0)$.

3 The Option of War in a Strategic Setting

3.1 Impact of Conflict

In a strategic context, however, Country $-i$ may have already taken possession of the resource, or may decide to attack Country i . We therefore need to make particular assumptions about the impact of the conflict situation on the costs. The cost C_i we considered earlier is only

applicable if the territory to be appropriated is not already occupied. If it is occupied, appropriating it requires waging war. We model war as a fixed cost that is applied instantaneously. Costs are suffered by both the attacker and the defender: W_i is the cost for Country i attacking Country $-i$, and D_i is the cost for Country i defending against Country $-i$. After the attacker wages war, both countries will share the resource, with an allocation θ_i for Country i and $\theta_{-i} = 1 - \theta_i$ for Country $-i$. After this war, the country which held the full territory loses a share of its benefit in exploiting the resource, as a consequence.

Assume Country i holds the territory and exploits the resource at time 0. Country $-i$ sets a threshold l_{-i} such that when it is reached, they will wage war. In these conditions, the strategy's value to Country $-i$ can be expressed in a similar way to what we have calculated earlier for $R_{l_{-i}}^i(S_0)$:

$$R_{l_{-i}}^{-i}(S_0) = -\alpha_{-i}S_0 + \left(\frac{S_0}{l_{-i}}\right)^\beta (\theta_{-i}\alpha_{-i}l_{-i} - W_{-i}).$$

The future gains in exploiting the resource, for Country $-i$, are only a fraction θ_{-i} of what they would be if they were the only actor.

From Country i 's perspective, who is exploiting the resource alone initially, the expected future costs of procuring the resource is 0 if Country $-i$ does not wage war. If it does, then it becomes:

$$R_{l_{-i}}^i(S_0) = -\left(\frac{S_0}{l_{-i}}\right)^\beta (\theta_i\alpha_i l_{-i} + D_i).$$

We can therefore write the value of the strategies (l_i, l_{-i}) before any of the countries takes the resource, in all generality for Country i as:

$$\begin{aligned} R_{l_i, l_{-i}}^i(S_0) = & \mathbb{1}_{l_i < l_{-i}} \left[-\alpha_i S_0 + \left(\frac{S_0}{l_i}\right)^\beta (\alpha_i l_i - C_i) - \left(\frac{S_0}{l_{-i}}\right)^\beta ((1 - \theta_i)\alpha_i l_{-i} + D_i) \right] \\ & + \mathbb{1}_{l_i \geq l_{-i}} \left[-\alpha_i S_0 + \left(\frac{S_0}{l_i}\right)^\beta (\theta_i \alpha_i l_i - W_i) \right]. \end{aligned}$$

There is a discontinuity when l_i and l_{-i} are equal: the first to invest has a cost $C + D$ while the other country has a cost W .

Given l_{-i} , one can determine the optimal level for $l_i \geq l_{-i}$ in a deterministic fashion. Indeed, Country i , in this case can find its optimal attack threshold with the only condition that it be greater than l_{-i} . Letting Country $-i$ take over the resource first, results in a value, for Country i , of:

$$-\alpha_i S_0 + \left(\frac{S_0}{l_i}\right)^\beta (\theta_i \alpha_i l_i - W_i).$$

The optimal war threshold without any constraint is therefore $T_i = \frac{W_i \beta}{\theta_i \alpha_i (\beta - 1)}$. However, since we must have $l_i \geq L_{-i}$, the optimal attack takes place when the underlying price reaches $L_{-i} \vee T_i$. If W_i is very low, and so is T_i , for example, then Country i should attack immediately after Country $-i$ has taken over the resource.

In this conflict situation, however, we must determine how the decision to be the first to take over the resource can be made.

3.2 Game-theoretical Equilibrium

Each country needs to determine an optimal strategy to set the l thresholds. If one country uses a relatively low threshold and the other one a high threshold, the first one will attack and occupy, but war should only take place in the distant future. If the thresholds are low, we expect to observe war soon, with high costs attached. We can see there is a strategic aspect, and that there is a game at play, roughly comparable to a game of chicken: the lower the threshold, the least cooperative the behavior. In order to find an optimal strategy, we assume that potentially different thresholds are drawn from an optimal probability distribution at the Bayes-Nash equilibrium. At the optimum, if a player is drawing from the optimal distribution, then whatever pure strategy the other player follows, should result in the same value to them. There is a timing effect in this situation, however: the first one to take possession of the territory gives the other country the ability to chose their optimal entry level unilaterally. Therefore, both can attempt to preempt one another with a randomized strategy, but once the first one has taken the territory, the optimal threshold for the other one becomes deterministic.

We write L_i and L_{-i} the independent random variables corresponding to the optimal strategic choice for each country. From the standpoint of Country i , we write the value of these strategies $R_{L_i, L_{-i}}^{*,i}(S_0)$, depending on the relative values of L_i and L_{-i} , as $R_{L_i, L_{-i} \geq}^{*,i}(S_0)$ or $R_{L_i, L_{-i} <}^{*,i}(S_0)$. If $L_i \geq L_{-i}$, then Country i can determine their optimal attack threshold with the only constraint that it be higher than L_{-i} . Therefore, in this case:

$$R_{L_i, L_{-i} \geq}^{*,i}(S_0) = -\alpha_i S_0 + \left(\frac{S_0}{L_{-i} \vee T_i} \right)^\beta (\theta_i \alpha_i (L_{-i} \vee T_i) - W_i).$$

If on the contrary $L_i < L_{-i}$, then i is the first to take over the territory once price L_i is reached, but then Country $-i$ determine their own optimal level, which is symmetrical to the prior

case, and equal to: $L_i \vee T_{-i}$. Therefore, in this case:

$$R_{L_i, L_{-i} <}^{*,i}(S_0) = -\alpha_i S_0 - \left(\frac{S_0}{L_i}\right)^\beta (\theta_i \alpha_i L_i - C_i) \\ - \left(\frac{S_0}{L_i \vee T_{-i}}\right)^\beta ((1 - \theta_i) \alpha_i (L_i \vee T_{-i}) + D_i).$$

At the optimum in a Bayes-Nash equilibrium, the strategy followed by $-i$, in setting the distribution of L_{-i} , should make Country i indifferent to any specific deterministic choice of L_i . Therefore, there exists a constant c independent of u such that:

$$c = \int_{S_0}^{\infty} \mathbb{P}[L_{-i} \in dv] R_{u,v}^{*,i}(S_0) \\ = \int_{S_0}^u \mathbb{P}[L_{-i} \in dv] R_{u,v,\geq}^{*,i}(S_0) + \int_u^{\infty} \mathbb{P}[L_{-i} \in dv] R_{u,v,<}^{*,i}(S_0) \\ = \int_{S_0}^u \mathbb{P}[L_{-i} \in dv] R_{u,v,\geq}^{*,i}(S_0) + \mathbb{P}[L_{-i} \geq u] R_{u,v,<}^{*,i}(S_0).$$

since $R_{u,v,<}^{*,i}(S_0)$ does not depend on v , and $R_{u,v,\geq}^{*,i}(S_0)$ does not depend on u . We write the density $\mathbb{P}[L_{-i} \in dv] = f_{L_{-i}}(v)dv$, which cannot depend on L_i . Taking the derivative of the above equation with respect to u , we obtain after some simplifications:

$$0 = f_{L_{-i}}(u) \left(R_{u,u,\geq}^{*,i}(S_0) - R_{u,v,<}^{*,i}(S_0) \right) + \frac{\partial}{\partial u} R_{u,v,<}^{*,i}(S_0) \mathbb{P}[L_{-i} \geq u].$$

We write $F_{L_{-i}}(u) = \mathbb{P}[L_{-i} \geq u]$, so that $F'_{L_{-i}} = -f_{L_{-i}}$. The equation can be rewritten:

$$F'_{L_{-i}}(u) = F_{L_{-i}}(u) \frac{\frac{\partial}{\partial u} R_{u,v,<}^{*,i}(S_0)}{R_{u,u,\geq}^{*,i}(S_0) - R_{u,v,<}^{*,i}(S_0)}.$$

We have:

$$R_{u,u,\geq}^{*,i}(S_0) = -\alpha_i S_0 + \left(\frac{S_0}{u \vee T_i}\right)^\beta (\theta_i \alpha_i (u \vee T_i) - W_i)$$

and:

$$R_{u,v,<}^{*,i}(S_0) = -\alpha_i S_0 - \left(\frac{S_0}{u}\right)^\beta (\theta_i \alpha_i u - C_i) \\ - \left(\frac{S_0}{u \vee T_{-i}}\right)^\beta ((1 - \theta_i) \alpha_i (u \vee T_{-i}) + D_i).$$

If we assume that $S_0 = 1$, without much loss in generality, we have:

$$\begin{aligned} R_{u,u,\geq}^{*,i} - R_{u,v,<}^{*,i} &= u^{-\beta} (\theta_i \alpha_i u - C_i) + (u \vee T_i)^{-\beta} (\theta_i \alpha_i (u \vee T_i) - W_i) \\ &\quad + (u \vee T_{-i})^\beta ((1 - \theta_i) \alpha_i (u \vee T_{-i}) + D_i) \\ \frac{\partial}{\partial u} R_{u,v,<}^{*,i} &= -u^{-\beta} \left(\theta_i \alpha_i (\beta - 1) - \frac{C_i}{u} + \mathbb{1}_{u < T_i} (1 - \theta_i) \alpha_i (\beta - 1) - \mathbb{1}_{u < T_{-i}} \frac{D_i}{u} \right). \end{aligned}$$

If the characteristics of both countries are the same, so that $T_i = T_{-i}$, these expressions can be simplified to some extent, and we can express the solution to the differential equation in a concise manner. We obtain the solution:

$$\mathbb{P}[L_{-i} \geq u] = \exp \left(- \int_1^u dx \frac{\theta_i \alpha_i (\beta - 1) - \frac{C_i}{x} + \mathbb{1}_{x < T_i} (1 - \theta_i) \alpha_i (\beta - 1) - \mathbb{1}_{x < T_i} \frac{D_i}{x}}{\alpha_i \theta_i x - C_i + x^\beta (x \vee T_i)^{-\beta} (\alpha_i (x \vee T_i) + D_i - W_i)} \right).$$

For x large (and greater than T_i in particular), the term under the integral in the exponential can be approximated by $\frac{\theta_i (\beta - 1)}{(\theta_i + 1)x}$, as x goes to infinity. Hence, we can see that when u goes to infinity:

$$\mathbb{P}[L_{-i} \geq u] \sim e^{-\frac{\theta_i (\beta - 1)}{\theta_i + 1} \ln(u)} = u^{-\frac{\theta_i (\beta - 1)}{\theta_i + 1}}.$$

This is a fat-tailed distribution, and more specifically a power law. The probability that one would have to wait until an arbitrary large threshold u is reached before a first country occupies the territory, and the other country then wages war at a higher threshold depending on this one, is therefore:

$$\mathbb{P}[L_i \geq u \cap L_{-i} \geq u] \sim u^{-\frac{2\theta_i (\beta - 1)}{\theta_i + 1}}.$$

The rate of decay of this probability is slow, so that there are chances that war may in fact take place in a distant future only. The Laplace transform of the war-trigger level $L_W = T_i \wedge T_{-i}$ (when a country first takes possession of the resource) hence verifies, for u large:

$$\mathbb{E} [e^{-\rho T_{LW}} | L_W \geq u] \sim u^{\gamma_i} \int_u^\infty dv \gamma_i v^{-\beta - \gamma_i - 1} = \frac{u^{-\beta}}{\beta + \gamma_i},$$

where we write $\gamma_i = \frac{2\theta_i (\beta - 1)}{\theta_i + 1}$. Note that $\mathbb{E} [e^{-\rho T_{LW}} | L_W \geq u] \sim \frac{1}{\beta + \gamma_i} \mathbb{E} [e^{-\rho T_u}]$. When both countries act optimally in the sense of game theory, the time until conflict, asymptotically, behaves in a comparable fashion to a simple hitting time, and decays relatively slowly. Therefore, we can see that, due to the strategic nature of the game here, and in particular in spite of both countries' propensity towards preemption, war may be indefinitely delayed.

References

- Acemoglu, Daron, Mikhail Golosov, Aleh Tsyvinski, & Pierre Yared.** 2012. "A Dynamic Theory of Resource Wars." *The Quarterly Journal of Economics* 127 (1): 283–331.
- Arasteh, Abdollah.** 2016. "Combination of Real Options and Game-Theoretic Approach in Investment Analysis." *Journal of Industrial Engineering International* 12 (3): 361–375.
- Azevedo, Alcino, & Dean Paxson.** 2014. "Developing Real Option Game Models." *European Journal of Operational Research* 237 (3): 909–920.
- Boulding, Kenneth E.** 1962. *Conflict and Defense: A General Theory*. New York: Harper Perennial.
- Caselli, Francesco, Massimo Morelli, & Dominic Rohner.** 2015. "The Geography of Interstate Resource Wars." *The Quarterly Journal of Economics* 130 (1): 267–315.
- Chesney, Marc, & Laurent Gauthier.** 2006. "American Parisian Options." *Finance and Stochastics* 10 (4): 475–506.
- D'Alpaos, Chiara, Michele Moretto, & Paolo Rosato.** 2023. "Common-Property Resource Exploitation: A Real Options Approach." *Land* 12 (7): 1304.
- Gauthier, Laurent.** 2002a. "Hedging Entry and Exit Decisions: Activating and Deactivating Barrier Options." *Journal of Applied Mathematics and Decision Sciences* 6 (1): 51–70.
- . 2002b. "Real Options and Exotic Options: A Probabilistic Approach." PhD Thesis, Paris: Université Panthéon-Sorbonne - Paris I. <https://tel.archives-ouvertes.fr/tel-00002076>.
- Grenadier, Steven R.** 2000. "Option Exercise Games: The Intersection of Real Options and Game Theory." *Journal of Applied Corporate Finance* 13 (2): 99–107.
- Hirshleifer, Jack.** 1988. "The Bioeconomic Causes of War." *Managerial and Decision Economics* 19 (7/8): 457–466.
- Jeanblanc, Monique, Marc Yor, & Marc Chesney.** 2009. *Mathematical Methods for Financial Markets*. Springer Finance Textbooks. London: Springer.
- Maoz, Zeev.** 1990. *Paradoxes of War: On the Art of National Self-Entrapment*. Studies in International Conflict 3. Boston: Unwin-Hyman.
- Martin, Philippe, Thierry Mayer, & Mathias Thoenig.** 2008. "Make Trade Not War?" *Review of Economic Studies* 75 (3): 865–900.
- Medina, Vincent, & Cyr-Denis Nidier.** 2003. "Pricing War Within a Real Option Framework." *Defence and Peace Economics* 14 (6): 425–435.
- Schelling, Thomas C.** (1960) 1980. *The Strategy of Conflict*. Cambridge: Harvard University Press.
- Wagner, R. Harrison.** 1983. "The Theory of Games and the Problem of International Cooperation." *American Political Science Review* 77 (2): 330–346.