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# Investment Under Uncertainty with Implementation Delay

Laurent GAUTHIER and Erwan MORELLEC\*

March 8, 2000

## Abstract

One of the major characteristics of the capital budgeting process is the delay existing between the investment decision and its implementation. This paper analyses investment decisions under uncertainty with implementation delay in a unified analytical framework. We provide closed-form solutions relating the value of the investment opportunity and the optimal investment threshold to the size of the delay. We show that the implementation lag creates an embedded option for the investor: the option to abandon the project during this delay. We derive the value of this option for various exercise policies corresponding to different levels of freedom with respect to the abandonment of the project and analyse its effects on the investment policy of the firm.

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Since the early 80's, advances in the real options literature have completely changed the way we evaluate investment opportunities. However, in spite of large theoretical advances, little research has focused on the practical side of the investment spending. One of the major characteristics of the capital budgeting process is the delay existing between the investment decision and its implementation. This implementation lag is generally associated with the construction period, the decision process within the firm or the gathering of the financing funds necessary to undertake the capital expenditure. Empirical studies by Ross (1986) and Taggart (1987) document for example that projects are generally initiated from the bottom up, suggesting a centralization of the capital allocation process. Depending on the nature and the size of the investment spending, projects that have been approved at the division level may have to be submitted to headquarters. Although all these intermediary steps may have important valuation consequences, they have generally been ignored in the real options literature<sup>1</sup>.

This paper analyses in a unifying framework the valuation effects of investment lags<sup>2</sup>. We derive closed-form formulas relating the value of the investment opportunity and the optimal investment policy to the size of the delay. We find that the implementation delay has a large negative impact on the value of the investment opportunity when the commitment to invest is irreversible. We then compute the value of the project when the firm has an abandonment option allowing it to unwind the investment decision during or after the implementation period, depending on the ongoing profitability of the investment project. From the simulations, we draw two main conclusions. First, European abandonment options, i.e. profitability requirements applying at the implementation date, do not increase the welfare of investors when the investment decision is made optimally. Second, "American" abandonment options, i.e. profitability requirements applying to the whole implementation period, partly offset the value loss associated with implementation lags.

In the following section, we build a framework allowing the valuation of investment projects when there is an implementation lag. Section two derives the value of investment projects for various investment policies. Sec-

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<sup>1</sup>Bar-Ilan and Strange (1996) extend Dixit's (1989) analysis to incorporate investment lags in a real options model. However they do not analyse the valuation effects of investment lags.

<sup>2</sup>Our analysis significantly differs from the sequential investment literature where the firm is not committed to complete the project. In this literature each dollar invested in the investment opportunity gives the firm the option to spend another dollar in the project. See for example Majd and Pindyck (1987) or Pindyck (1993).

tion three concludes the paper and presents possible extensions. Technical developments are gathered in the appendix.

## 1 A general model of investment decisions with implementation delay

This section develops the valuation framework. In order to emphasize the effect of lags on investment decisions, we consider the simplest possible setting. As a result, the model relies on the following assumption. (A1) Markets are perfect with no transaction costs. (A2) There exists a constant riskfree rate  $r$  at which investors may lend and borrow freely. (A3) The firm has an exclusive access to a project producing output forever<sup>3</sup>. The constant capital expenditure is denoted by  $C$ , production costs are zero and the production rate is one unit of output per unit of time. (A4) There exists a portfolio of traded assets that pays dividends at the constant rate  $\delta$  and the return of which is perfectly correlated with the output price. (A5) Under the unique risk neutral probability measure  $Q$ , the dynamics of the output price  $(S_t)_{t \geq 0}$  are given by

$$\frac{dS_t}{S_t} = (r - \delta) dt + \sigma dW_t \quad (1)$$

where  $\sigma$  is a constant parameter and  $(W_t)_{t \geq 0}$  is a standard Brownian motion on the probability space  $(\Omega, \mathcal{F}, Q)$ . In equation (1),  $\delta$  can be interpreted as a constant convenience yield.

The Markovian features of the model and the stationarity of the distribution of the payoffs generated by the active project imply that the investment decision occurs at the first instant when the decision variable  $(S_t)_{t \geq 0}$  hits some *constant* upper optimal barrier  $h^*$ . One common characteristic of traditional real options models<sup>4</sup> is the implicit assumption that actions are taken instantaneously: The project starts producing output as soon as the investor has decided to invest. Therefore, the flow of profits accrues to investors at the stopping time  $T_{h^*}(S)$  defined by

$$T_{h^*}(S) = \inf \{t \geq 0, S_t = h^*\}$$

In this paper, we evaluate investment projects when there is a delay between the investment decision and its implementation. As mentioned

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<sup>3</sup>Standard justifications of the assumption of irreversibility rely on the lemons problem or capital specificity of the assets in place (see for example Abel and alii (1996)). The irreversibility assumption is very realistic for economic activities which are highly capital intensive such as mining projects or offshore petroleum leases.

<sup>4</sup>See for example Dixit and Pindyck (1994) for a review.

earlier, this implementation delay can be due to the research of an investment opportunity on an emerging market, to the capital budgeting process within the firm, to the construction lag or to the time spent gathering the financing funds. When we take the implementation lag into account, the project starts producing output at a stopping time  $T_{h^*}(S) + \theta(S)$  where  $\theta(S)$  is a parametrized time independent of  $T_{h^*}(S)$  and such that the level of the decision variable at time  $T_{h^*}(S) + \theta(S)$  is independent of  $\theta(S)$  and  $T_{h^*}(S)$ . This added time can be viewed as a general constraint. It can be a fixed time or any time that conditions the implementation of the investment spending.

Suppose that the firm invests at an arbitrary investment boundary  $h$ . The value of the investment project for  $S_0 < h$  is given by

$$V(h, \Delta, C, \theta) = \mathbb{E}_{S_0}^Q \left[ \int_{T_h(S)+\theta(S)}^{\infty} dt e^{-rt} S_t \right] - C \mathbb{E}_{S_0}^Q \left[ e^{-r(T_h(S)+\theta(S))} \right] \quad (2)$$

In equation (2), the first term of the RHS is the expected present value of the profits generated by the project whereas the second term represent the expected present value of the capital expenditure. Using standard results of Brownian motion calculus, we obtain after simplifications

$$V(h, \Delta, C, \theta) = \left( \frac{S_0}{h} \right)^\xi \mathbb{E}_h^Q \left[ e^{-r\theta(S)} \right] \left( \Delta h \mathbb{E}_h^Q \left[ \frac{S_{\theta(S)}}{h} \right] - C \right) \quad (3)$$

where  $\Delta = \delta^{-1}$  and  $\xi = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\frac{2r}{\sigma^2} + \left( \frac{1}{2} - \frac{r-\delta}{\sigma^2} \right)^2}$ .

Let us denote  $A(\theta) = \mathbb{E}_h^Q \left[ e^{-r\theta(S)} \right]$  and  $B(\theta) = \mathbb{E}_h^Q \left[ S_{\theta(S)}/h \right]$ .  $A(\theta)$  is the discounting factor associated with the implementation delay while  $B(\theta)$  accounts for the change in the expected operating profit due to the path of the decision variable during this period. Using equation (3), we can write

$$V(h, \Delta, C, \theta) = V(h, \Delta A(\theta) B(\theta), CA(\theta), 0) \quad (4)$$

The value of the investment opportunity, for a given investment barrier  $h$ , can be expressed as the value of the investment opportunity with no delay and modified parameters. Straightforward calculations give us the optimal barrier and the optimal value of the investment opportunity.

**Proposition 1** *When there is an implementation delay, the value of the investment opportunity and the optimal investment threshold are respectively given by*

$$V(h^*, \Delta, C, \theta) = A(\theta) \left( \frac{\Delta B(\theta) S_0}{\xi} \right)^\xi \left( \frac{C}{\xi - 1} \right)^{1-\xi} \quad (5)$$

and

$$h^*(\Delta, C, \theta) = \frac{\xi}{\xi - 1} \frac{C}{\Delta B(\theta)} \quad (6)$$

The following section shows that the implementation delay reduces the value of investment opportunities and provides alternative investment rules aimed at minimizing the associated value reduction.

## 2 Implementation delay and embedded options

### 2.1 Implementation delay with no exit option

Let us consider first the case where the delay is a fixed time  $d$  and the firm invests at time  $T_h(S) + d$  whatever the evolution of the decision variable during the time interval  $[T_h(S), T_h(S) + d]$ . This simple case gives us the value reduction associated with the implementation lag when there is no abandonment option i.e. when the commitment to invest is irreversible.

Straightforward calculations yield

$$A(\theta) = e^{-rd}, \quad B(\theta) = e^{(r-\delta)d} \quad \text{and} \quad h^*(\Delta, C, d) = e^{-(r-\delta)d} h^*(\Delta, C, 0)$$

When there is an implementation delay, investors anticipate the investment decision if they anticipate a positive growth of the state variable or postpone it otherwise.

Using equation (5), we see that the ratio of the value of the investment opportunity with delay to that with no delay is

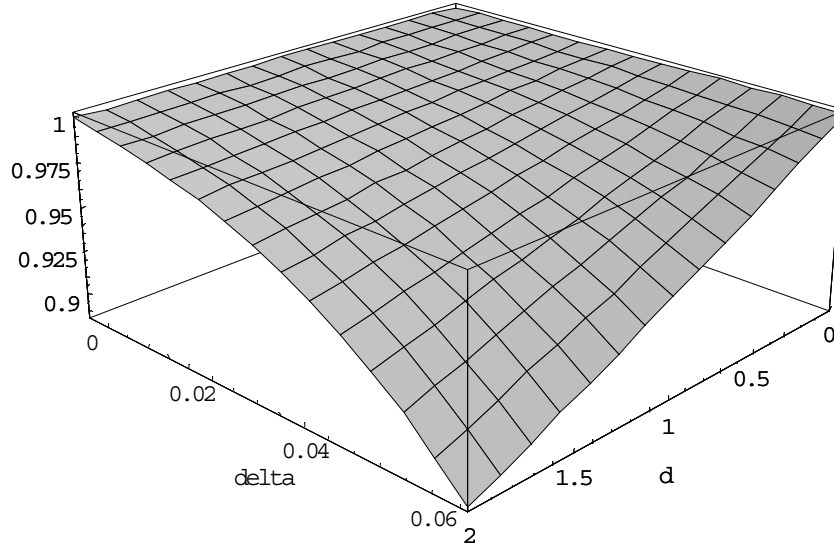
$$r(\theta) = \exp\{(\xi(r-\delta) - r)d\} < 1$$

Figure 1 below plots  $r(\theta)$  as a function  $d$  and  $\delta$  for  $d \in [0, 2]$ ,  $\delta \in [0, 0.065]$ ,  $r = 0.07$  and  $\sigma = 0.2$ . It reveals that the implementation lag has a significant impact on the value of the project. The associated value reduction is due to the discounting of the investment NPV from  $T_h + d$  to  $T_h$  and to the dependence of the value of the project on the evolution of the decision variable during the implementation period<sup>5</sup>. Also, observe that the value reduction due to the implementation delay is highly sensitive to the

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<sup>5</sup>The average change in the decision variable during the implementation delay is given by  $\exp((r-\delta)d)$  which is larger than one for  $r > \delta$ . However, abandonment options are not valueless since the real path followed by the decision variable can be unfavorable to the firm.

value of the convenience yield: As the opportunity cost of remaining *inactive* increases, the value loss due to the implementation delay gets larger.



## 2.2 European abandonment option

We consider in this section that the firm has a European abandonment option allowing it to unwind the investment decision at the end of the implementation lag. The investment decision is still triggered when the decision variable reaches some constant upper barrier  $h$ . However, it is implemented only if the state variable is above a new cutoff level  $h'$  at the end of the implementation lag.

Consider for example, a decision taken at division level that has to be approved by the headquarters. If the approval arrives  $d$  units of time after the submittance, then the operational manager invests only if the state variable is above a new cutoff level  $h'$ . The headquarters have no own profitability requirements during the implementation lag and the approval is based on the strategical merits of the project (new line of business, expanded markets). By contrast, the operational manager has profitability requirements each time he is in charge with the decision i.e. at the initiation of the project and at the end of the implementation period.

In this new setting, the project starts producing output at the first time when  $d$  units of time after having hit  $h$ , the state variable is above  $h'$ . This stopping time is mathematically described in the appendix. Using

proposition 3 from the appendix, we get

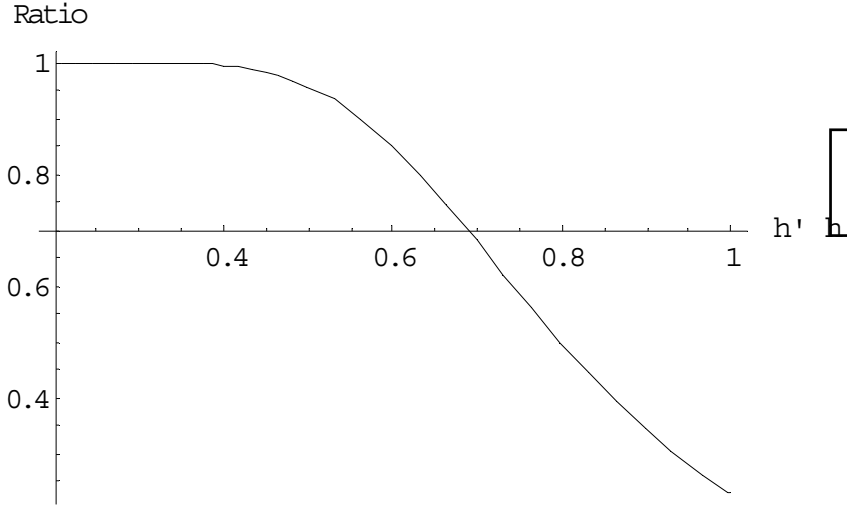
$$A(\theta) = e^{-rd} \frac{1 - \mathcal{N}\left(b\sqrt{d} + \ln(h'/h) (\sigma\sqrt{d})^{-1}\right)}{1 - \mathcal{N}\left(-\sqrt{(2r + b^2)d} + \ln(h'/h) (\sigma\sqrt{d})^{-1}\right)} \quad (7)$$

$$B(\theta) = e^{(r-\delta)d} \frac{1 - \mathcal{N}\left((b + \sigma)\sqrt{d} + \ln(h'/h) (\sigma\sqrt{d})^{-1}\right)}{1 - \mathcal{N}\left(-b\sqrt{d} + \ln(h'/h) (\sigma\sqrt{d})^{-1}\right)}. \quad (8)$$

where  $b = (r - \delta - \sigma^2/2) / \sigma$  and  $\mathcal{N}$  is the Standard Normal cumulative distribution function

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt, x \in \mathbb{R}$$

Figure 2 below plots the ratio of the value of the investment opportunity with the European abandonment option to that with no abandonment option.



The European abandonment option will never be exercised as we have  $h'/h = 0$  at the optimum. When the manager holds a European abandonment option, the value of waiting to invest at the maturity of this security (i.e. at the implementation date) is lower than the benefits of investing



directly. Postponing the investment spending at the end of the implementation period reduces the value of the project by a larger amount than an immediate exercise does.

Let us consider for example the simple special case where  $h' = h$ . Then we have from equations (7) and (8)

$$A(\theta) = e^{-rd} \frac{1 - \mathcal{N}(b\sqrt{d})}{1 - \mathcal{N}(-\sqrt{(2r+b^2)d})} \quad \text{and} \quad B(\theta) = e^{(r-\delta)d} \frac{1 - \mathcal{N}((b+\sigma)\sqrt{d})}{1 - \mathcal{N}(-b\sqrt{d})}$$

The ratio of the value of the investment opportunity with the European abandonment option to that with no abandonment option satisfies

$$r(\theta) = \frac{1 - \mathcal{N}(b\sqrt{d})}{1 - \mathcal{N}(-\sqrt{(2r+b^2)d})} \left( \frac{1 - \mathcal{N}((b+\sigma)\sqrt{d})}{1 - \mathcal{N}(-b\sqrt{d})} \right)^\xi$$

which is lower than 1.

### 2.3 The Parisian abandonment option

This section provides the value of the project when the manager invests at the end of the implementation delay only the decision variable remains above the investment threshold during this whole period. This investment criterion reflects the willingness of the firm to check that market conditions remain favorable during the implementation lag.

In order to find the value of the investment opportunity under this so-called investment Parisian<sup>6</sup> policy, we define the following random variables, all linked to a random process  $S$ :

$$\begin{aligned} g_t^h(S) &= \sup \{s \leq t : S_s = h\} \\ \tau_d^h(S) &= \inf \left\{ t \geq 0 : \left( t - g_t^h(S) \right) \geq d, S_t \geq h \right\}. \end{aligned}$$

$g_t^h(S)$  represents the last time before  $t$  the process  $S$  crossed the level  $h$ .  $\tau_d^h(S)$  is therefore the first instant when the process has spent  $d$  units of time consecutively over the level  $h$ .

Using section 2 of the appendix, we get

$$A(\theta) = \frac{\Phi(b\sqrt{d})}{\left( \Phi(\sqrt{d(2r+b^2)}) \right)} \quad \text{and} \quad B(\theta) = \frac{\Phi((\sigma+b)\sqrt{d})}{\left( \Phi(b\sqrt{d}) \right)}$$

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<sup>6</sup>See Chesney, Jeanblanc and Yor (1997) for a definition of Parisian options.

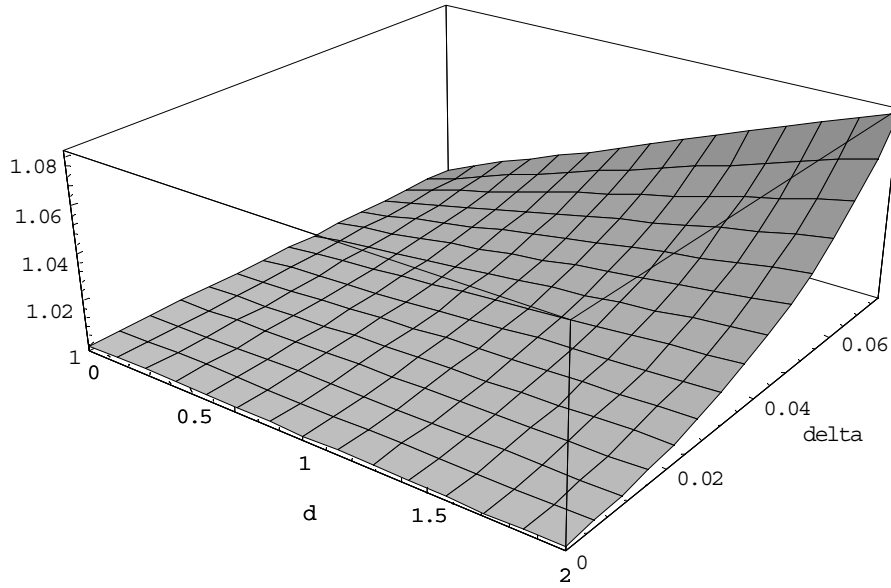
where

$$\Phi(x) = 1 + \sqrt{2\pi}x \exp(x^2/2) \mathcal{N}(x).$$

The ratio of the value of the project when the manager holds a Parisian option to that of the standard case with an implementation delay is

$$r(\theta) = \frac{e^{d[r-\xi(r-\delta)]} \Phi(b\sqrt{d})}{\Phi(\sqrt{d(2r+b^2)})} \left( \frac{\Delta \Phi((\sigma+b)\sqrt{d})}{\xi \Phi(b\sqrt{d})} \right)^\xi$$

Figure 3 below plots  $r(\theta)$  as a function  $d$  and  $\delta$  for  $d \in [0, 2]$ ,  $\delta \in [0, 0.065]$ ,  $r = 0.07$  and  $\sigma = 0.2$ . The value of the investment opportunity is higher when the investor has the opportunity to choose the Parisian investment criterion. This is particularly true when the opportunity cost of remaining inactive (represented by  $\delta$ ) is large.



When the investing firm holds a Parisian abandonment option, the optimal investment boundary is given by

$$h(\Delta, C_e, d) = h(\Delta, C_e, 0) \frac{\Phi(b\sqrt{d})}{\Phi((\sigma+b)\sqrt{d})}$$

Since  $\Phi(\cdot)$  is a strictly increasing function, the investment boundary is lower than in the standard case with no implementation delay. Moreover, the larger the implementation lag, the lower the optimal investment barrier is and, thus, the lower the hurdle rate used by the firm.

The Parisian investment policy provides the value of the decision variable under which there will be no investment. There is a value of waiting to invest but the value of the state variable at the investment date and the option premium can vary according to the basic parameters and the shape of the excursion of the decision variable above the barrier. The value of the decision variable for which investment will occur and the value of the option premium can therefore be over or under the standard ones.

## 2.4 Exponential abandonment policy

Standard results concerning American options show that the optimal exercise barrier exhibits some time dependence. Therefore, the Parisian criterion is not absolutely optimal since the abandonment barrier is constant.

In this section, we give the value of the investment opportunity when the manager invests only if the decision variable remains during the implementation delay above a time dependent early abandonment boundary. Following Omberg (1987) and Ju (1998), we approximate the early exercise boundary of the American abandonment option by an exponential function which is given the following form  $g(t) = he^{-\sigma(\epsilon+\beta t)}$  where  $\epsilon$  and  $\beta$  are positive real numbers.  $\epsilon > 0$  implies that there is an initial jump in the abandonment barrier while  $\beta$  takes into account the time dependence of this boundary. According to this specification, the firm takes into account the cost of postponing once again the investment spending and has a decreasing minimum profitability requirement during the whole implementation lag.

Proposition 5 gives the value of the coefficients  $A(\theta)$  and  $B(\theta)$  when the manager follows this exponential exercise policy. We have

$$\begin{aligned} A(\theta) &= \mathbb{E}_0 \left[ e^{-\left(r+\frac{b^2}{2}\right)\eta_d^{y,0}(Z)} \right] \mathbb{E}_0 \left[ e^{\frac{bZ}{\eta_d^{y,0}}} \right] \\ B(\theta) &= \mathbb{E}_0 \left[ e^{-\frac{b^2}{2}\eta_d^{y,0}(Z)} \right] \mathbb{E}_0 \left[ e^{(b+\sigma)Z} \eta_d^{y,0}(Z) \right] \end{aligned}$$

and for all positive  $\lambda$  and positive  $\alpha$

$$\mathbb{E} \left[ e^{-\lambda\eta_d^{y,0}} \right] = \frac{e^{-\lambda d} \int_d^\infty dt \frac{\epsilon \exp\left(-\frac{(\epsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}}{1 - e^{-\epsilon\sqrt{2\lambda}} \int_0^d dt \frac{\epsilon \exp\left(-(\lambda+\beta\sqrt{2\lambda})t - \frac{(\epsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}}$$

$$\mathbb{E} \left[ e^{\alpha Z_{\tau_d}^{y,0}} \right] = e^{-\alpha\beta d - \frac{\beta^2}{2}d} \frac{\int_{-\varepsilon}^0 dv \int_v^\infty du \frac{2(2u-v)}{\sqrt{2\pi d^3}} e^{-\frac{(2u-v)^2}{2t} + (\alpha+\beta)u}}{\int_d^\infty dt \frac{\varepsilon \exp\left(-\frac{(\varepsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}}.$$

The optimal investment boundary is then obtained by maximizing the value of the investment opportunity over  $h' = h e^{-\sigma\varepsilon}$  and  $\beta$ .

Numerical simulations indicate that the abandonment option with exponential exercise barrier has a larger value than the Parisian abandonment option. This is what was expected as we have now two degrees of freedom  $h'$  and  $\beta$  instead of one. This stresses that there exists a time dependence of the optimal abandonment level to the time already spent in the approval stage or to how long the firm has been consecutively gathering investors.

### 3 Concluding remarks

This paper provides a valuation framework for investment opportunities relying on the computation of first passage times. We show that the delay existing between the investment decision and its implementation has important valuation consequences. In particular, when the investing firm has no profitability requirement once the investment decision has been made, the value of the investment opportunity can be reduced by almost 10% for a one year delay. We also show that European abandonment option do not increase the value of the investment project. By contrast, the so-called Parisian and exponential abandonment policies partly offset the value loss associated with the implementation lag.

Beyond the analysis of the effects of the capital budgeting practices within firms, typical applications of our model include the services offered by specialized investment funds or the cost of the recourse to outside financing. For large projects it often takes considerable time to gather all the financing funds as the decision process within the institutions involved in the project can be highly time consuming. For example, the time necessary to gather investors for a closed-end investment fund typically reaches one year during which the economic and market conditions can completely change. The type of financing funds may have an impact over the investment policy of firms as they condition the availability of the option to abandon investment opportunities during the implementation delay. In the same way, the services offered by specialized investment funds constitute a typical application of these options. Investment opportunities on emerging markets can take time to be realized and an investment specialist can reduce the delay between the

investment decision and its real implementation by providing a dedicated vehicle. The best example in that case is an open-ended fund. The price to pay for the immediacy of the opportunity is reflected in the bid-ask spread for the fund as opposed to a closed-end investment in which case the monies cannot be withdrawn.

Finally, one should mention that the approach presented in this paper is general enough to allow us to extend the scope of our analysis to other rigidities in the investment process. We focused in this paper on the delay existing between the investment decision and its real implementation. Other imperfections or rigidities can be considered such as noise existing in the information available to the investor concerning the profitability of his investment opportunity or the competition for corporate resources due to the capital allocation process within firms.

## 4 Appendix

### 4.1 The European abandonment option

Let us define  $\nu_d^{-a,0}$ , the first instant when,  $d$  units of time after reaching 0, the Brownian Motion is above  $-a$ . To be able to formally define  $\nu_d^{-a,0}$ , we create the series of random times  $T^{(i)}$ .  $T^{(0)} = d$ . If the Brownian Motion is above  $-a$  at that time, then  $\nu_d^{-a,0} = T^{(0)} = d$ . If not, we look at the first instant after  $T^{(0)}$  when the Brownian Motion reaches 0, that is  $T^{T^{(0)},0} = T^*(1)$ . Then, we write  $T^{(1)} = T^*(1) + d$ , that is  $T^{(1)}$  is  $d$  units of time after the Brownian Motion has reached zero. If the Brownian Motion is above  $-a$  at  $T^{(1)}$ , and if it was below  $-a$  in  $T^{(0)}$  then  $\nu_d^{-a,0} = T^{(1)}$ . Following the same procedure one can generate the series of  $T^{(i)}$ . We have  $T^{T^{(1)},0} = T^*(2)$  and  $T^{(2)} = T^*(2) + d$ .

Now we define

$$\begin{aligned}\widehat{T}^{(0)} &= T^{(0)}\mathbb{I}_{B_{T^{(0)}} \geq -a} \\ \widehat{T}^{(1)} &= T^{(1)}\mathbb{I}_{B_{T^{(1)}} \geq -a}\mathbb{I}_{\widehat{T}^{(0)}=0} \text{ and generally} \\ \widehat{T}^{(i)} &= T^{(i)}\mathbb{I}_{B_{T^{(i)}} \geq -a} \prod_{k=1}^i \mathbb{I}_{\widehat{T}^{(i-k)}=0}\end{aligned}$$

Finally, we can write  $\nu_d^{-a,0} = \sum_i \widehat{T}^{(i)}$ . In this sum, the  $\widehat{T}^{(i)}$  are not independent. Indeed  $\mathbb{P}(\widehat{T}^{(1)} \neq 0, \widehat{T}^{(2)} \neq 0) = 0$  because of the characteristic function in the expression of  $\widehat{T}^{(2)}$ , while  $\mathbb{P}(\widehat{T}^{(1)} \neq 0)$  and  $\mathbb{P}(\widehat{T}^{(2)} \neq 0)$  are both strictly positive.

**Proposition 2** For all positive  $\lambda$  and positive  $\alpha$ ,  $\nu_d^{-a,0}$  and  $B_{\nu_d^{-a,0}}$  are independent and

$$\begin{aligned}\mathbb{E}\left[e^{-\lambda\nu_d^{-a,0}}\right] &= e^{-\lambda d} \frac{1 - \mathcal{N}\left(-a/\sqrt{d}\right)}{1 - \mathcal{N}\left(-\sqrt{2\lambda d} - a/\sqrt{d}\right)} \\ \mathbb{E}\left[e^{\alpha B_{\nu_d^{-a,0}}}\right] &= e^{d\frac{\alpha^2}{2}} \frac{1 - \mathcal{N}\left(\alpha\sqrt{d} - a/\sqrt{d}\right)}{1 - \mathcal{N}\left(-a/\sqrt{d}\right)}.\end{aligned}$$

**Proof.** The proof of this Proposition is based on the Markov property of Brownian Motion. Let us first focus on the Laplace transform of the law of  $\nu_d^{-a,0}$ . We have

$$\mathbb{E}\left[e^{-\lambda\nu_d^{-a,0}}\right] = \mathbb{E}\left[e^{-\lambda\sum_i \widehat{T}^{(i)}}\right]$$

Using the independence the Markov property we can write

$$\begin{aligned}
& \mathbb{E} \left[ e^{-\lambda \sum_i \widehat{T}^{(i)}} \right] \\
&= e^{-\lambda d} \sum_{i=1}^{\infty} \mathbb{E} \left[ \mathbb{I}_{B_d > -a} e^{-\lambda d} \right] \prod_{k=1}^i \mathbb{E} \left[ \mathbb{I}_{B_d \leq -a} e^{-\lambda(d+T^{d,0})} \right] + \mathbb{E} \left[ \mathbb{I}_{B_d > -a} e^{-\lambda d} \right] \\
&= e^{-\lambda d} \mathbb{E} \left[ \mathbb{I}_{B_d > -a} \right] \left( \sum_{i=1}^{\infty} \left( \mathbb{E} \left[ \mathbb{I}_{B_d \leq -a} e^{-\lambda(d+T^{d,0})} \right] \right)^i + 1 \right) \\
&= e^{-\lambda d} \frac{\mathbb{E} \left[ \mathbb{I}_{B_d > -a} \right]}{1 - \mathbb{E} \left[ \mathbb{I}_{B_d \leq -a} e^{-\lambda(d+T^{d,0})} \right]}.
\end{aligned}$$

Now, straightforward calculations give

$$\mathbb{E} \left[ \mathbb{I}_{B_d > -a} \right] = 1 - \mathcal{N} \left( -a/\sqrt{d} \right)$$

and

$$\begin{aligned}
\mathbb{E} \left[ \mathbb{I}_{B_d \leq -a} e^{-\lambda(d+T^{d,0})} \right] &= e^{-\lambda d} \mathbb{E} \left[ \mathbb{I}_{B_d \leq -a} e^{-|B_d| \sqrt{2\lambda}} \right] \\
&= \mathcal{N} \left( -\sqrt{2\lambda d} - a/\sqrt{d} \right).
\end{aligned}$$

Thus, we obtain

$$\mathbb{E} \left[ e^{-\lambda \nu_d^{-a,0}} \right] = e^{-\lambda d} \frac{1 - \mathcal{N} \left( -a/\sqrt{d} \right)}{1 - \mathcal{N} \left( \sqrt{2\lambda d} - a/\sqrt{d} \right)}.$$

Let us now focus on the position of the Brownian Motion. We can first notice, using the Markov property, that the process is only conditioned by the fact it was at the level 0  $d$  units of time before  $\nu_d^{-a,0}$ , and that it is higher than  $-a$ . Therefore, we just have to compute

$$\begin{aligned}
\mathbb{E} \left[ e^{\alpha B_{\nu_d^{-a,0}}} \right] &= \mathbb{E} \left[ e^{\alpha B_d} \mid B_d \geq -a \right] = \frac{\mathbb{E} \left[ e^{\alpha B_d} \mathbb{I}_{B_d \geq -a} \right]}{\mathbb{P} \left( B_d \geq -a \right)} \\
&= e^{d \frac{\alpha^2}{2}} \frac{1 - \mathcal{N} \left( \alpha \sqrt{d} - a/\sqrt{d} \right)}{1 - \mathcal{N} \left( -a/\sqrt{d} \right)} \blacksquare
\end{aligned}$$

The result can be extended to a drifted Brownian Motion:

**Proposition 3** For a positive  $\rho$  and for  $S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma Z_t\right)$  where  $Z$  is a Brownian Motion, we have

$$\mathbb{E}_h \left[ e^{-\rho \nu_d^{h',h}(S)} \right] = \frac{e^{-\rho d} \left( 1 - \mathcal{N} \left( b\sqrt{d} + \ln \left( \frac{h'}{h} \right) \left( \sigma\sqrt{d} \right)^{-1} \right) \right)}{1 - \mathcal{N} \left( -\sqrt{(2\rho + b^2)d} + \ln \left( \frac{h'}{h} \right) \left( \sigma\sqrt{d} \right)^{-1} \right)}$$

$$\mathbb{E}_h \left[ S_{\nu_d^{h',h}(S)}^\gamma \right] = h^\gamma e^{d\gamma\left(\mu - \frac{\sigma^2}{2}(1-\gamma)\right)} \frac{1 - \mathcal{N} \left( (b + \gamma\sigma)\sqrt{d} + \ln \left( \frac{h'}{h} \right) \left( \sigma\sqrt{d} \right)^{-1} \right)}{1 - \mathcal{N} \left( -b\sqrt{d} + \ln \left( \frac{h'}{h} \right) \left( \sigma\sqrt{d} \right)^{-1} \right)}.$$

**Proof.** This result is obtained thanks to a simple application of Girsanov's theorem to Proposition 2 using the following Radon Nikodym derivative

$$\forall t \geq 0, \frac{d\mathbb{P}}{d\mathbb{L}} \Big|_{\mathcal{F}_t} = \exp(bB_t - b^2t/2) = \mathcal{L}_t. \blacksquare$$

## 4.2 The Parisian abandonment option

### 4.2.1 Laplace transforms

**Theorem 1 (Chesney Jeanblanc Yor 1995)** If  $B$  is a Brownian motion starting from zero, then

$$\forall \lambda \geq 0, \mathbb{E}[\exp(-\lambda\tau_d(B))] = \frac{1}{\Phi(\sqrt{2\lambda d})}$$

$$\text{if } \Phi(x) = \int_0^{+\infty} z \exp\left(zx - \frac{z^2}{2}\right) dz = 1 + \sqrt{2\pi}xe^{-\frac{x^2}{2}} \mathcal{N}(x).$$

**Proof.** We follow the lines of Chesney, Jeanblanc and Yor (1995) and start with some definitions. The result could be proved by the use of excursion theory and the description of Itô's measure<sup>7</sup>, but the approach detailed here, originally due to Chesney, Jeanblanc and Yor (1995), is less demanding.

**Definition 1** We define the so-called slow filtration  $\mathcal{G} = (\sigma(\text{sgn}(B_t)) \vee \mathcal{F}_{g_t})_{t \geq 0}$ .

It represents the information on the Brownian motion until its last zero plus the knowledge of its sign after this. We intend to project a certain martingale onto this new filtration.

<sup>7</sup>See for example Revuz and Yor (1991) exercise (4.10)



**Definition 2** We will call Azéma's martingale the process

$$(\mu_t = \operatorname{sgn}(B_t) \sqrt{t - g_t}, t \geq 0)$$

One can easily check that we have<sup>8</sup>  $\mu_t = \sqrt{\frac{\pi}{2}} \mathbb{E}[B_t / \mathcal{G}_t]$ .

**Definition 3** If we fix a real number  $t$ , the Brownian meander is the process

$$\left( m_u = \frac{1}{\sqrt{t - g_t}} |B_{g_t + u(t - g_t)}|, 1 \geq u \geq 0 \right).$$

This meander is obviously independent from the variable  $\operatorname{sgn}(B_t)$  and both are independent from the  $\sigma$ -algebra  $\mathcal{F}_{g_t}$ . Following Chung (1976) we give a few hints on how these results are derived.

First, let us see how the joint law  $(B_t, g_t)$  is determined. Using the so-called *reflection principle* of Désiré André, we can write

$$\mathbb{P}_x(B_t \in dy, T_0 > t) = \frac{1}{\sqrt{2\pi t}} \left( e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}} \right) dy.$$

Now, if  $0 < s < t$ ,  $x > 0$ ,  $y > 0$ ,

$$\begin{aligned} & \mathbb{P}(g_t \leq s, |B_s| \in dx, |B_t| \in dy) \\ &= \mathbb{P}(|B_s| \in dx, \forall u \in [s, t], |B_u| \neq 0, |B_t| \in dy) \\ &= \mathbb{P}(|B_s| \in dx) \mathbb{P}(\forall u \in [s, t], |B_u| \neq 0, |B_t| \in dy \mid |B_s| = x) \\ &= \mathbb{P}(|B_s| \in dx) \mathbb{P}_x(|B_{t-s}|, T_0 > t - s). \end{aligned}$$

The last equality is obtained thanks to the Markov property. The explicit form of this equation is known, and by integration with respect to  $x$  and derivation with respect to  $s$  we get

$$\mathbb{P}(g_t \in ds, |B_t| \in dy) = ds dy \frac{\mathbb{I}_{0 < s < t} \mathbb{I}_{y > 0} y e^{-\frac{y^2}{2(t-s)}}}{\pi \sqrt{s(t-s)}}.$$

Thanks to this last result, one gets:

$$\mathbb{P}(m_1 \in dz) = \mathbb{P}\left(\frac{|B_t|}{\sqrt{t - g_t}} \in dz\right) = \mathbb{I}_{z \in \mathbb{R}^+} z \exp\left(-\frac{z^2}{2}\right) dz.$$

---

<sup>8</sup> A study of various properties of this martingale, along with other projections on the slow filtration is performed in Azéma and Yor (1989).

As for the independence of the meander with the Brownian Motion up to its last zero, one method is to write the law of the meander as a limit in finite distributions. That is to show that for  $0 < u_1 < \dots < u_n < 1$

$$\mathbb{P}(m_{u_1} \in dx_1, \dots, m_{u_n} \in dx_n)$$

does not depend on  $t$  and is independent of  $g_t$ . Let us write it for  $m_1$ . We have

$$\begin{aligned} \mathbb{P}(m_1 \in dx, g_1 \in ds) &= \mathbb{P}\left(\frac{|B_t|}{\sqrt{t-g_t}} \in dx, g_1 \in ds\right) \\ &= \mathbb{P}\left(\frac{|B_t|}{\sqrt{t-g_t}} \in dx\right) \mathbb{P}(g_1 \in ds) \end{aligned}$$

We shall now turn to the proof the theorem. Let us define the following exponential martingale:

$$M_t^\lambda = \exp\left(bB_t - \frac{b^2t}{2}\right).$$

To prove the theorem, we choose to focus first on another stopping time:

$$\tau_d(B) = \inf\{0 \leq s : s - g_s(B) = d, B_s \geq 0\}.$$

Now, following Azéma and Yor (1989), let us consider  $(\widetilde{M}_t^\lambda, t \geq 0)$  the projection of the  $\mathcal{F}$ -martingale  $(M_t^\lambda, t \geq 0)$  on  $\mathcal{G}$ , defined by

$$\widetilde{M}_t^\lambda = \mathbb{E}\left[M_t^\lambda \mid \mathcal{G}_t\right]$$

Then, we have

$$\widetilde{M}_t^\lambda = \Phi(\lambda\mu_t) \exp\left(-\frac{\lambda^2}{2}t\right)$$

To show this, notice that  $B_t = m_1\mu_t$ , with  $m_1$  and  $\mu_t$  independent. Then, let us write

$$\begin{aligned} \widetilde{M}_t^\lambda &= \mathbb{E}\left[\exp\left(\lambda B_t - \frac{\lambda^2}{2}t\right) \mid \mathcal{G}_t\right] \\ &= \exp\left(-\frac{\lambda^2}{2}t\right) \mathbb{E}[\exp(\lambda m_1 \mu_t) \mid \mathcal{G}_t] \\ &= \exp\left(-\frac{\lambda^2}{2}t\right) \int_{\mathbb{R}^+} \exp(\lambda z \mu_t) \mathbb{P}(m_1 \in dz) \end{aligned}$$

the last equality coming from the fact  $\mu_t$  is  $\mathcal{G}_t$ -measurable. Writing that expression explicitly gives

$$\begin{aligned}\widetilde{M}_t^\lambda &= \exp\left(-\frac{\lambda^2}{2}t\right) \int_{\mathbb{R}^+} \exp(\lambda z \mu_t) z \exp\left(-\frac{z^2}{2}\right) dz \\ &= \Phi(\lambda \mu_t) \exp\left(-\frac{\lambda^2}{2}t\right)\end{aligned}$$

It is useful to note that  $(\widetilde{M}_{t \wedge \tau_d(B)}^\lambda, t \geq 0)$  is a bounded martingale. Indeed, it is positive and  $\widetilde{M}_{t \wedge \tau_d(B)}^\lambda \leq \Phi(\lambda d)$ . Now, we can apply the optional stopping theorem to the martingale  $(\widetilde{M}_{t \wedge \tau_d(B)}^\lambda, t \geq 0)$ , since  $\tau_d(B)$  is almost surely a finite stopping time of the filtration  $\mathcal{G}$ :

$$\mathbb{E}\left[\widetilde{M}_{t \wedge \tau_d(B)}^\lambda\right] = 1$$

Thanks to Lebesgue's theorem, we now let  $t$  go to infinity and obtain for any  $\lambda \geq 0$

$$\begin{aligned}1 &= \mathbb{E}\left[\widetilde{M}_{\tau_d(B)}^\lambda\right] \\ 1 &= \mathbb{E}\left[\Phi\left(\lambda \mu_{\tau_d(B)}\right) \exp\left(-\frac{\lambda^2}{2}\tau_d(B)\right)\right] \\ \frac{1}{\Phi(\lambda \sqrt{d})} &= \mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}\tau_d(B)\right)\right]\end{aligned}$$

Therefore, we have  $\forall \lambda \geq 0$ ,

$$\frac{1}{\Phi(\lambda \sqrt{d})} = \mathbb{E}\left[\exp\left(-\frac{\lambda^2}{2}\tau_d(B)\right)\right]$$

And then

$$\mathbb{E}\left[\exp(-\lambda \tau_d^-(B))\right] = \frac{1}{\Phi(\sqrt{2\lambda d})} \blacksquare$$

The result can be extended to a drifted Brownian Motion:

**Theorem 2** *If  $Z_t = B_t + bt$ , the Laplace transform of  $\tau_d(Z)$  is given by*

$$\forall \lambda \geq 0, \mathbb{E}\left[\exp(-\lambda \tau_d(Z))\right] = \frac{\Phi(b\sqrt{d})}{\Phi(\sqrt{(2\lambda + b^2)d})}.$$

**Proof.** Using Girsanov's theorem, we can write for  $\lambda \geq 0$

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[\exp(-\lambda\tau_d(Z))] &= \mathbb{E}_{\mathbb{L}}\left[\exp(-\lambda\tau_d(Z))\frac{d\mathbb{P}}{d\mathbb{L}}\Big|\mathcal{F}_{\tau_d(Z)}\right] \\ &= \mathbb{E}_{\mathbb{L}}\left[\exp(-\lambda\tau_d(Z))\exp\left(bB_{\tau_d(Z)} - \frac{b^2}{2}\tau_d(Z)\right)\right] \\ &= \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\left(\lambda + \frac{b^2}{2}\right)\tau_d(B)\right)\exp(bB_{\tau_d(B)})\right]\end{aligned}$$

Moreover we know that  $\tau_d$  and  $B_{\tau_d}$  are independent, and in law,  $B_{\tau_d} = m_d = \sqrt{d}m_1$ . Therefore

$$\mathbb{E}_{\mathbb{P}}[\exp(-\lambda\tau_d(Z))] = \mathbb{E}_{\mathbb{P}}\left[e^{-(\lambda + \frac{b^2}{2})\tau_d(B)}\right] \int_0^{+\infty} e^{bx\frac{x}{d}} \exp\left(-\frac{x^2}{2d}\right) dx.$$

After some simplifications, we get for any  $\lambda \geq 0$

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[\exp(-\lambda\tau_d(Z))] &= \frac{1}{\Phi\left(\sqrt{(2\lambda + b^2)d}\right)} \int_0^{+\infty} e^{bx\frac{x}{d}} \exp\left(-\frac{x^2}{2d}\right) dx \\ &= \frac{\Phi(b\sqrt{d})}{\Phi\left(\sqrt{(2\lambda + b^2)d}\right)} \blacksquare\end{aligned}$$

**Theorem 3** *If  $Z_t = B_t + bt$ , we have the following Laplace transform*

$$\forall \lambda \geq 0, \mathbb{E}_{\mathbb{P}}[\exp(-\lambda\tau_d^a(Z))] = \exp\left(ba - |a|\sqrt{2\lambda + b^2}\right) \frac{\Phi(b\sqrt{d})}{\Phi\left(\sqrt{(2\lambda + b^2)d}\right)}.$$

**Proof.** We write that

$$\tau_d^a(Z) = T_a(Z) + \tau_d(Z \circ \theta_{T_a(Z)} - a)$$

where  $\theta$  is the so-called "shift operator" on the canonical space  $\Omega$ . Using the strong Markov property and the fact that the trajectories after and before  $T_a$  are independent, we can write

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[\exp(-\lambda\tau_d^a(Z))] &= \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}[\exp(-\lambda\tau_d^a(Z))|\mathcal{F}_{T_a(Z)}]\right] \\ &= \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}[\exp(-\lambda(T_a(Z) + \tau_d(Z \circ \theta_{T_a(Z)} - a))|T_a(Z))]\right] \\ &= \mathbb{E}_{\mathbb{P}}[\exp(-\lambda T_a(Z))]\mathbb{E}_{\mathbb{P}}[\exp(-\lambda\tau_d(Z))] \blacksquare\end{aligned}$$

### 4.2.2 Value of the investment opportunity

Under the Parisian investment policy, the value of the investment opportunity is given by

$$V(h, \Delta, C, \theta) = \mathbb{E}_{S_0}^Q \left[ e^{-r\tau_d^h(S)} \left( F \left( S_{\tau_d^h(S)}, \infty \right) - C_e \right) \right]$$

where  $F(S_t, \infty)$  is the expected present value of future profits accruing to claimholders when the investment spending is realized at time  $t$  i.e.

$$F(S_t, \infty) = \int_t^\infty ds e^{-\rho(s-t)} \mathbb{E}_{S_t} [f(S_s)]$$

Writing  $S_t = S_0 \exp(\sigma Z_t^b)$  where  $Z_t^b = B_t + bt$  and  $b = (\mu - \sigma^2/2)/\sigma$ , we have

$$V(h, \Delta, C, \theta) = \mathbb{E}_{S_0}^Q \left[ e^{-r\tau_d^a(Z^b)} \left( F \left( S_0 \exp \left( \sigma Z_{\tau_d^a(Z^b)}^b \right), \infty \right) - C_e \right) \right]$$

with  $a = \ln(h/S_0)/\sigma$ . Thanks to the equality in law between  $\tau_d^a(Z^b)$  and  $\tau_d^0(Z^b) + T^a(Z^b)$  for two independent copies, we get

$$V(h, \Delta, C, \theta) = \left( \frac{S_0}{h} \right)^\xi \mathbb{E}_{S_0}^Q \left[ e^{-r\tau_d^0(Z^b)} \right] \left( \mathbb{E}^Q \left[ F \left( h e^{\sigma\sqrt{d}m_1}, \infty \right) \right] - C_e \right)$$

where  $m_1$  is the Brownian meander taken at time 1. Finally, using theorem 2 and after simplifications, we obtain

$$V(h, \Delta, C, \theta) = \left( \frac{S_0}{h} \right)^\xi \frac{\Phi(b\sqrt{d})}{\Phi(\sqrt{d}(2\rho + b^2))} \left[ \Delta h \frac{\Phi((\sigma + b)\sqrt{d})}{\Phi(b\sqrt{d})} - C_e \right].$$

### 4.3 The exponential abandonment barrier

We start by defining, for a given measurable negative function  $y$ , the following stopping times

$$H^{t,y} = \inf \{s \geq t : B_s = y(s-t)\}.$$

We define the stopping time  $\eta_d^{y,0}$ , the first instant when the Brownian motion spends more than  $d$  units of time over  $y$ , the countdown starting when the Brownian motion reaches  $y$ . As an approximation of the optimal policy, we are interested in a particular  $y$ , of the form

$$y(t) = -\varepsilon - \beta t.$$

We have then the following result

**Proposition 4** For all positive  $\lambda$  and positive  $\alpha$ ,  $\eta_d^{y,0}$  and  $B_{\eta_d^{y,0}}$  are independent and

$$\begin{aligned}\mathbb{E}^Q \left[ e^{-\lambda \eta_d^{y,0}} \right] &= \frac{e^{-\lambda d} \int_d^\infty dt \frac{\varepsilon \exp\left(-\frac{(\varepsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}}{1 - e^{-\varepsilon\sqrt{2\lambda}} \int_0^d dt \frac{\varepsilon \exp\left(-(\lambda+\beta\sqrt{2\lambda})t - \frac{(\varepsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}} \\ \mathbb{E}^Q \left[ e^{\alpha B_{\eta_d^{y,0}}} \right] &= e^{-\alpha\beta d - \frac{\beta^2}{2}d} \frac{\int_{-\varepsilon}^0 dv \int_v^\infty du \frac{2(2u-v)}{\sqrt{2\pi d^3}} e^{-\frac{(2u-v)^2}{2t} + (\alpha+\beta)u}}{\int_d^\infty dt \frac{\varepsilon \exp\left(-\frac{(\varepsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}}.\end{aligned}$$

**Proof.** We shall first concentrate on the Laplace transform of the law of  $\eta_d^{y,0}$ . We can write directly

$$\begin{aligned}\mathbb{E}^Q \left[ e^{-\lambda \eta_d^{y,0}} \right] &= \sum_{i=0}^{\infty} \left( \mathbb{E}^Q \left[ \mathbb{I}_{H^{0,y} \leq d} e^{-\lambda(H^{0,y} + H^{H^{0,y},0})} \right] \right)^i \mathbb{E}^Q \left[ \mathbb{I}_{H^{0,y} > d} e^{-\lambda d} \right] \\ &= \frac{\mathbb{E}^Q \left[ \mathbb{I}_{H^{0,y} > d} e^{-\lambda d} \right]}{1 - \mathbb{E}^Q \left[ \mathbb{I}_{H^{0,y} \leq d} e^{-\lambda(H^{0,y} + H^{H^{0,y},0})} \right]}.\end{aligned}$$

Now, we use the fact that for a Brownian motion  $B$ ,

$$H^{0,y}(B) = \inf \{t \geq 0 : B_t + \beta t = -\varepsilon\} = T_{-\varepsilon}(B^\beta).$$

We also know that  $B_{H^{0,y}} = -\varepsilon - \beta H^{0,y}$ . Furthermore, since the law of  $H^{H^{0,y},0}$  only depends on  $H^{0,y}$  through the position of the Brownian motion at that time, we get in law

$$H^{H^{0,y},0} = T_{-\varepsilon - \beta H^{0,y}}.$$

These considerations allow us to write

$$\mathbb{E}^Q \left[ \mathbb{I}_{H^{0,y} \leq d} e^{-\lambda(H^{0,y} + H^{H^{0,y},0})} \right] = e^{-\varepsilon\sqrt{2\lambda}} \int_0^d dt \frac{\varepsilon \exp\left(-(\lambda + \beta\sqrt{2\lambda})t - \frac{(\varepsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}$$

On the other hand, we get easily

$$\mathbb{E}^Q \left[ \mathbb{I}_{H^{0,y} > d} e^{-\lambda d} \right] = e^{-\lambda d} \int_d^\infty dt \frac{\varepsilon \exp\left(-\frac{(\varepsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}.$$

This last results completes the computation.

Let us now consider the position of the Brownian motion. The Markov property of the process ensures that its position at the time  $\eta_d^{y,0}$  is only conditioned by the fact that  $d$  units of time before it was at the level 0, and that it has not hit the barrier during this period. Therefore we just have to compute

$$\begin{aligned}
\mathbb{E}^Q \left[ e^{\alpha B_{\eta_d^{y,0}}} \right] &= \mathbb{E}^Q \left[ e^{\alpha B_d} \mid T_{-\varepsilon}(B^\beta) \geq d \right] \\
&= e^{-\alpha\beta d} \frac{\mathbb{E}^Q \left[ e^{\alpha B_d^\beta} \mathbb{I}_{\inf_{u \leq d} B_u \geq -\varepsilon} \right]}{Q(T_{-\varepsilon}(B^\beta) \geq d)} \\
&= e^{-\alpha\beta d - \frac{\beta^2}{2}d} \frac{\int_{-\varepsilon}^0 dv \int_v^\infty du \frac{2(2u-v)}{\sqrt{2\pi d^3}} e^{-\frac{(2u-v)^2}{2t} + (\alpha+\beta)u}}{\int_d^\infty dt \frac{\varepsilon \exp\left(-\frac{(\varepsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}}.
\end{aligned}$$

where the last equality uses the joint law between the Brownian motion and its minimum. ■

We are now interested in similar results for the geometric Brownian motion.

**Proposition 5** *For a positive  $r$  and for  $S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma Z_t\right)$  where  $Z$  is a Brownian motion, and  $g(t) = he^{-\sigma(\varepsilon+\beta t)}$  we have*

$$\begin{aligned}
\mathbb{E}_h^Q \left[ e^{-r\eta_d^{g,h}(S)} \right] &= \mathbb{E}^Q \left[ e^{-\left(r + \frac{b^2}{2}\right)\eta_d^{y,0}(Z)} \right] \mathbb{E}^Q \left[ e^{bZ_{\eta_d^{y,0}}} \right] \\
\mathbb{E}_h^Q \left[ S_{\eta_d^{g,h}(S)} \right] &= h \mathbb{E}^Q \left[ \exp\left((b + \sigma)Z_{\eta_d^{y,0}(Z)}\right) \right] \mathbb{E}^Q \left[ \exp\left(-\frac{b^2}{2}\eta_d^{y,0}(Z)\right) \right]
\end{aligned}$$

**Proof.** This result is obtained by applying Girsanov's theorem to proposition 4. ■

## References

- [1] Abel, Andrew, Avinash Dixit, Janice Eberly, and Robert Pindyck, 1996, "Options, the Value of Capital, and Investment", *Quarterly Journal of Economics* 111, 753-777.
- [2] Azéma, Jean and Marc Yor, 1989, "Etude d'une Martingale Remarquable", *Séminaire de Probabilité XXIII*, Lecture Notes in Mathematics 1372, Springer-Verlag.
- [3] Bar-Ilan, Avner and William Strange, 1996, "Investment Lags", *American Economic Review* 86, 610-622.
- [4] Chesney Marc, Monique Jeanblanc and Marc Yor, 1995, "Brownian Excursions and Parisian Barrier Options", *Advances in Applied Probabilities* 29, 165-184.
- [5] Chung, Kai Lai, 1976, "Excursions in Brownian Motion", *Arkiv för Matematik*, 14, 155-177.
- [6] Dixit, Avinash, 1989, "Entry and Exit Decisions under Uncertainty", *Journal of Political Economy* 97, 620-638.
- [7] Dixit, Avinash and Robert Pindyck, 1994, *Investment Under Uncertainty*, Princeton, NJ: Princeton University Press.
- [8] Ju, Nengjiu, 1998, "Pricing an American Option by Approximating Its Early Exercise Boundary as a Multipiece Exponential Function", *Review of Financial Studies* 11, 627-646
- [9] Majd, Saman and Robert Pindyck, 1987, "Time to Build, Option Value, and Investment Decisions", *Journal of Financial Economics* 18, 7-27.
- [10] Omberg, Edward, 1987, "The Valuation of American Put Options with Exponential Exercise Policies", *Advances in Futures and Options Research* 2, 117-142.
- [11] Pindyck, Robert, 1993, "Investments of Uncertain Cost", *Journal of Financial Economics* 34, 53-76.
- [12] Revuz Daniel and Marc Yor, 1991, *Continuous Martingales and Brownian Motion*, Springer-Verlag, Berlin Heidelberg New York.
- [13] Ross, Marc, 1986, "Capital Budgeting Practices of Twelve Large Manufacturers", *Financial Management* 15, 15-22.



- [14] Taggart, Robert, 1987, "Allocating Capital Among a Firm's Division: Hurdle Rate vs. Budgets", *Journal of Financial Research* 10, 177-190.